An asymptotic solution for slightly buoyant laminar plumes

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When the buoyancy forces are small compared with the inertia forces, heated plumes in laminar flows which are uniform at upstream infinity approximately satisfy a linearized version of the Boussinesq equations, here called the Oseen–Boussinesq equations. An analytic solution is constructed for arbitrary Prandtl number and arbitrary direction of the unperturbed flow in the case of a plume produced by a point source. The two-dimensional case of the plume from a line source is considered briefly. A Stokes-type paradox occurs: it is found that a line-source solution that vanishes at infinity does not exist.

1. Introduction

In this paper an asymptotic approximation is derived for slightly buoyant plumes in a laminar flow which is uniform at upstream infinity. The inclination to the direction of gravity of the oncoming flow is arbitrary, as is the Prandtl number. By slightly buoyant plumes we mean plumes in which the buoyancy forces are an order of magnitude smaller than the inertia forces. For such plumes the Boussinesq equations can be approximated by a linear set of equations, which we shall refer to as the Oseen–Boussinesq equations. It will be shown that these equations give rise to a Stokes-type paradox. That is to say, for a two-dimensional plume from a line source in an infinite region, a solution that vanishes at infinity does not exist, whereas for a three-dimensional plume from a point source such a solution does exist; this solution will be constructed explicitly.

2. The Oseen limit of the Boussinesq equations

The equations governing steady buoyant laminar plumes, if the density does not vary too much, are the Boussinesq equations:

$$(\mathbf{u}.\nabla)\,\mathbf{u} = -\rho^{-1}\nabla p + \nu\nabla^2\mathbf{u} + \beta g\theta(\mathbf{e}_x\sin\alpha + \mathbf{e}_y\cos\alpha),$$
$$(\mathbf{u}.\nabla)\,\theta = \kappa\nabla^2\theta, \quad \operatorname{div}\mathbf{u} = 0. \tag{2.1}$$

The y axis makes an angle α in the anticlockwise direction with the vertical, $\mathbf{e}_x$ and $\mathbf{e}_y$ are unit vectors in the x and y directions, β is the thermal expansion coefficient, g the acceleration due to gravity, θ the difference between the temperatures inside and outside the plume, ρ the density, p the pressure, $\mathbf{u}$ the velocity and ν and κ the kinematic viscosity and diffusivity, respectively. At

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upstream infinity the flow is assumed to be uniform and in the x direction. Either a point or a line source of heat is assumed to be present at the origin. The source of the plume is assumed to produce no mass or momentum. The flow takes place in an unbounded region.

L, U, ρ and $\Theta = Q(\rho U L^\gamma c_p)^{-1}$ are chosen as the units of length, velocity, density and temperature, where U is the velocity of the flow at upstream infinity, Q the rate of heat production from a unit length of the line source in the two-dimensional case and from the point source in the three-dimensional case, c_p the specific heat at constant pressure and γ equals 1 or 2 in two or three dimensions, respectively. The length unit L is left unspecified and can be chosen arbitrarily.

If dimensionless quantities are denoted by the same symbols as the corresponding dimensional quantities, (2.1) become in dimensionless form

$$\left.\begin{array}{l}(\mathbf{u}.\nabla)\,\mathbf{u} = -\nabla p + (2\lambda)^{-1}\nabla^2\mathbf{u} + \epsilon\theta(\mathbf{e}_x\sin\alpha + \mathbf{e}_y\cos\alpha), \\ (\mathbf{u}.\nabla)\,\theta = (2\lambda\sigma)^{-1}\nabla^2\theta, \quad \operatorname{div}\mathbf{u} = 0.\end{array}\right\} \tag{2.2}$$

Here $\lambda = UL/(2\nu)$ is half the Reynolds number, $\epsilon = g\beta Q(\rho U^3 L^{\gamma-1}c_p)^{-1}$ and $\sigma = \nu/\kappa$ is the Prandtl number.

The dimensionless number ϵ† is a measure of the ratio of the magnitudes of the buoyancy and the inertia forces, and equals $Ra\,(4\sigma\lambda^2)^{-1}$, where

$$Ra = g\beta QL^{3-\gamma}\,(\nu\kappa\sigma U c_p)^{-1}$$

is the Rayleigh or Grashof number. Furthermore, ϵ equals the square of what is known as the buoyancy frequency in the theory of waves in stratified fluids, made non-dimensional with the time unit L/U.

The boundary conditions are

$$u = 1, \quad v = w = p = \theta = 0 \quad \text{at 'infinity'}, \tag{2.3}$$

where 'infinity' excludes a parabolic region downstream of the source.

For $\epsilon = 0$ the solution for the flow field is

$$u = 1, \quad v = w = p = 0. \tag{2.4}$$

For ϵ small, the solution may be expected to deviate little from (2.4), so the Oseen linearization may be applied. The following asymptotic series are introduced:

$$\left.\begin{array}{l}\theta = \theta_0 + \epsilon\theta_1 + ..., \quad \mathbf{u} = \mathbf{e}_x + \epsilon\mathbf{u}_1 + ..., \\ p = \epsilon p_1 + \epsilon^2 p_2 + ...\end{array}\right\} \tag{2.5}$$

The parameter ϵ can be made small in various ways; for example by choosing the rate of heat production of the source to be small. In three dimensions the arbitrary length scale L enters ϵ, and $\epsilon \ll 1$ if $L \gg g\alpha Q(\rho U^3 c_p)^{-1}$; hence the following analysis is valid at distances much greater than $O[g\alpha Q(\rho U^3 c_p)^{-1}]$ from the source in the three-dimensional case.

† There does not seem to be a commonly accepted name for ϵ. Ostrach (1964) called ϵ a Froude number because, like the original Froude number U^2/gL, it is a measure of the ratio of the body and inertia forces. However, because body forces may arise from a variety of causes, as for instance in the case of the Rossby and Alfvén numbers, it may lead to confusion always to call this ratio a Froude number. It has been suggested (Miles 1969) that ϵ should be named after John Scott Russell, who made observations on surface waves in the 19th century. Miles noted that this would make the principal parameters of geophysical fluid mechanics a 'vowel-ordered set': Ra, Re, Ri, Ro, Ru.

Substitution of (2.5) into (2.2) leads to the following system of equations for θ_0, $\mathbf{u}_1$ and p_1:

$$\partial\theta_0/\partial x = (2\lambda\sigma)^{-1}\nabla^2\theta_0, \tag{2.6}$$

$$\partial\mathbf{u}_1/\partial x = -\nabla p_1 + (2\lambda)^{-1}\nabla^2\mathbf{u}_1 + \theta_0(\mathbf{e}_x\sin\alpha + \mathbf{e}_y\cos\alpha), \tag{2.7}$$

$$\operatorname{div}\mathbf{u}_1 = 0. \tag{2.8}$$

With $\theta_0 \equiv 0$, (2.7) is the Oseen equation. We shall refer to the system (2.6)–(2.8) as the Oseen–Boussinesq equations.

3. The three-dimensional point-source solution of the Oseen–Boussinesq equations

Equation (2.6) is equivalent to the Helmholtz equation, as may be seen by making the substitution $\theta_0 = \psi(x,y,z)\exp(\lambda\sigma x)$. Its point-source solution is

$$\theta_0 = (\lambda\sigma/2\pi r)\exp\{\lambda\sigma(x-r)\}, \tag{3.1}$$

where $r^2 = x^2 + y^2 + z^2$.

The Green's function for the Oseen equation is well known. The solution of (2.7) is a convolution integral of this Green's function with θ_0. However, because of its complexity, this representation of the solution is rather unwieldy. The solution is obtained directly in a simple form by a method which is an adaptation of Lamb's (1911, 1932) elegant method for solving the Oseen equation for the flow around a circular cylinder or a sphere.

Taking the curl of (2.7) one obtains

$$\partial\mathbf{\gamma}_1/\partial x = (2\lambda)^{-1}\nabla^2\mathbf{\gamma}_1 + \nabla\theta_0 \times \mathbf{e}_g, \tag{3.2}$$

where $\mathbf{\gamma}_1 = \operatorname{curl}\mathbf{u}_1$ and $\mathbf{e}_g$ is a unit vector in the vertical direction. Equation (3.2) is satisfied by $\mathbf{\gamma}_1 = \nabla\chi_1 \times \mathbf{e}_g$, with the scalar χ_1 satisfying

$$\partial\chi_1/\partial x = (2\lambda)^{-1}\nabla^2\chi_1 + \theta_0. \tag{3.3}$$

By postulating a solution of the form $\chi_1 = f(x-r)$, the appropriate solution of (3.3) is found to be

$$\chi_1 = \begin{cases} [\lambda\sigma/2\pi(\sigma-1)][E_1\{\lambda(r-x)\} - E_1\{\lambda\sigma(r-x)\}], & \sigma \neq 1, \\ (\lambda/2\pi)\exp\{\lambda(x-r)\}, & \sigma = 1, \end{cases} \tag{3.4}$$

where
$$E_1(z) = \int_z^\infty \frac{1}{s}\exp(-s)\,ds.$$

Taking the divergence of (2.7) and using (2.8) one obtains

$$\nabla^2 p_1 = \frac{\partial\theta_0}{\partial x}\sin\alpha + \frac{\partial\theta_0}{\partial y}\cos\alpha. \tag{3.5}$$

The solution of this equation is very much simplified by the fact that a particular solution p_{1p} can be found from the following simple equation:

$$\frac{\partial p_{1p}}{\partial x} = \frac{1}{2\lambda\sigma}\left(\frac{\partial\theta_0}{\partial x}\sin\alpha + \frac{\partial\theta_0}{\partial y}\cos\alpha\right). \tag{3.6}$$

Equation (3.6) holds because, if p_{1p} satisfies (3.6) and (2.3), p_{1p}, like θ_0, is a solution of (2.6), so that solutions of (3.6) satisfy (3.5). With the help of (3.6), the solution of (3.5) is easily found to be

$$p_1 = \frac{\cos\alpha}{4\pi}[1 - \exp\{\lambda\sigma(x-r)\}]\frac{\partial}{\partial y}[\ln(r-x)] + \frac{\sin\alpha}{4\pi r}[\exp\{\lambda\sigma(x-r)\} - 1]. \quad (3.7)$$

The velocity field has the following form:

$$\mathbf{u}_1 = \nabla\phi_1 + \chi_1\mathbf{e}_g. \quad (3.8)$$

It remains to determine ϕ_1.

Substitution of (3.8) into (2.7) results in

$$\partial\phi_1/\partial x = -p_1 + (2\lambda)^{-1}\nabla^2\phi_1. \quad (3.9)$$

Because of (2.8), ϕ_1 must also satisfy

$$\nabla^2\phi_1 = -\sin(\alpha)\,\partial\chi_1/\partial x - \cos(\alpha)\,\partial\chi_1/\partial y. \quad (3.10)$$

Tentatively substituting (3.10) into (3.9) one obtains the following simple equation:

$$\phi = \int_{-\infty}^{x}\left\{-p_1 - (2\lambda)^{-1}\left(\sin\alpha\frac{\partial\chi_1}{\partial x} + \cos\alpha\frac{\partial\chi_1}{\partial y}\right)\right\}dx. \quad (3.11)$$

The complete solution will have been obtained if it can be shown that ϕ_1 as defined by (3.11) also satisfies (3.10). This is easily seen to be the case by applying the Laplace operator to (3.11) and taking (3.5) and (3.3) into account. The solution of (3.11) is

$$\phi_1 = \frac{y\cos\alpha}{4\pi(\sigma-1)}[\sigma(r-x)^{-1}\exp\{\lambda(x-r)\} - (r-x)^{-1}\exp\{\lambda\sigma(x-r)\}$$
$$- (\sigma-1)(r-x)^{-1} + \lambda\sigma E_1\{\lambda\sigma(r-x)\} - \lambda\sigma E_1\{\lambda(r-x)\}]$$
$$+ \frac{\sin\alpha}{4\pi(\sigma-1)}[E_1\{\lambda\sigma(r-x)\} - \sigma E_1\{\lambda(r-x)\} + (1-\sigma)\ln(r-x)] \quad (3.12a)$$

when $\sigma \neq 1$ and

$$\phi_1 = \frac{y\cos\alpha}{4\pi(r-x)}[\exp\{\lambda(x-r)\} - 1] - \frac{\sin\alpha}{4\pi}[E_1\{\lambda(r-x)\} + \ln(r-x) + \exp\{\lambda(x-r)\}]$$
$$(3.12b)$$

when $\sigma = 1$.

For the special case $\sigma = 1$, $\alpha = 0$ the limit of this solution as the Reynolds number goes to infinity has been obtained by Csanady (1965).

Because the potential of a horseshoe vortex is $y/(r-x)$, one may expect the flow pattern to resemble a horseshoe-vortex flow pattern. Behaviour of horse-shoe-vortex type of hot smokestack plumes can be observed visually under favourable circumstances, and is well documented.

Figures 1 and 2 give an impression of the flow pattern viewed in the direction of the x axis. By v, w curves we mean curves that are everywhere tangential to the vector $(0, v, w)$. For $x > 0$ the figures clearly reveal the horseshoe-vortex behaviour of the flow. For $\alpha = 0$ and $x > 0$ the v, w curves are apparently closed, for $\alpha \neq 0$ apparently not. For $x > 0$ and small the determination of those portions of the v, w curves that almost coincide with the y axis is numerically an ill-posed problem; therefore these portions have not been drawn. Because the
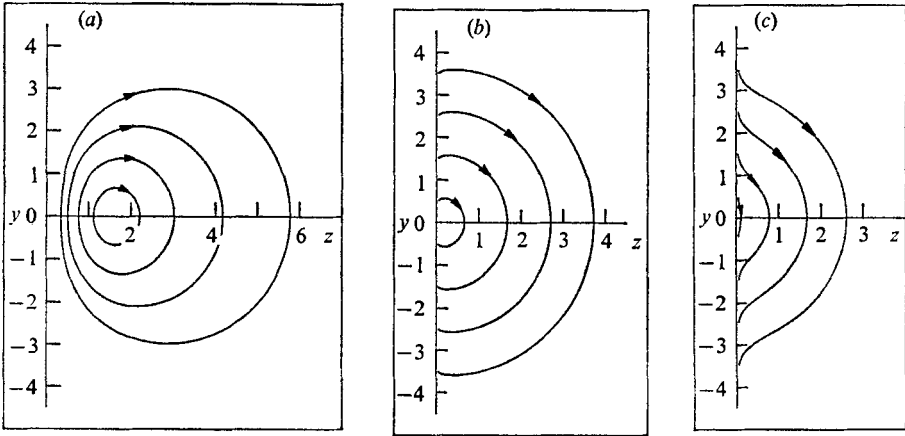
FIGURE 1. Three-dimensional plume in horizontal unperturbed flow: v, w curves with $\epsilon = 0\cdot1, \lambda = 10, \sigma = 1.$ $(a)\,x = 10, (b)\,x = 0\cdot05, (c)\,x = -0\cdot2.$
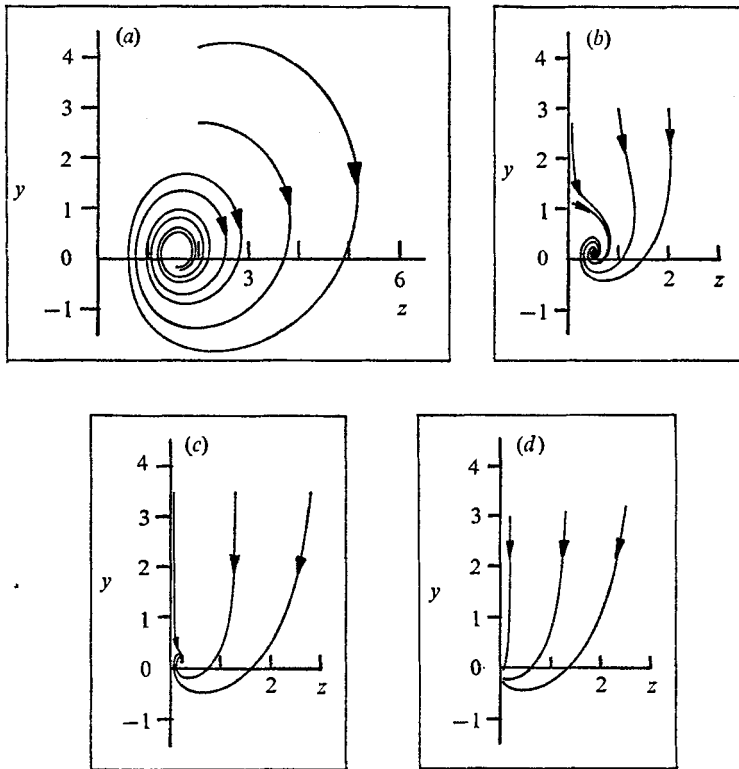

FIGURE 2. Three-dimensional plume in inclined unperturbed flow: v, w curves with $\alpha = 45°$, $\epsilon = 0\cdot1, \lambda = 10, \sigma = 1.$ $(a)\,x = 10, (b)\,x = 1, (c)\,x = 0\cdot2, (d)\,x = -0\cdot2.$

value of σ is found not to have much influence on the shape of the v, w curves, only one value of σ is represented. Figure 3 shows streamlines in the plane of symmetry when $\alpha = 0$. As σ increases the heat remains more concentrated, which results in a steeper rise of the streamlines that pass close to the source.
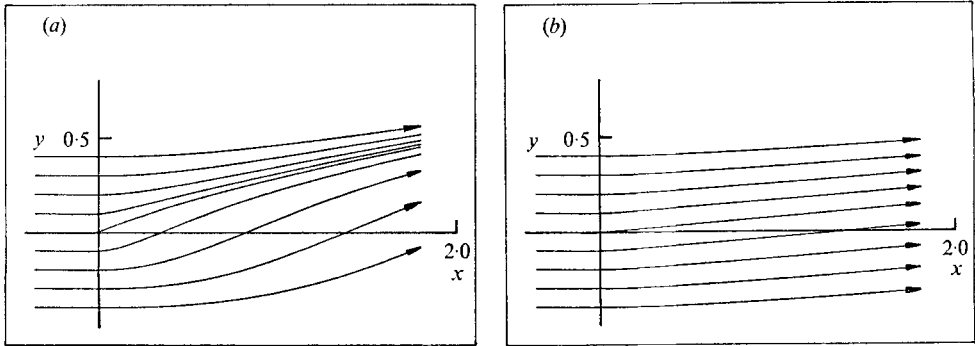
FIGURE 3. Streamlines in plane of symmetry for a three-dimensional plume with $\alpha = 0$, $\epsilon = 0.1$, $\lambda = 10$. (a) $\sigma = 3$, (b) $\sigma = 1$.

4. The two-dimensional line-source solution of the Oseen–Boussinesq equations

The line-source solution for θ_0 is

$$\theta_0 = (\lambda\sigma/\pi)\exp(\lambda\sigma x)K_0(\lambda\sigma r), \tag{4.1}$$

where $r = (x^2 + y^2)^{\frac{1}{2}}$ and K_0 is a modified Bessel function of the second kind, as defined by Watson (1958, p. 78).

The pressure satisfies (3.5), and a particular solution p_{1p} can be found from (3.6). Hence

$$p_{1p} = \frac{\sin\alpha}{2\pi}\exp(\lambda\sigma x)K_0(\lambda\sigma r) - \frac{\lambda\sigma y\cos\alpha}{2\pi}\int_{-\infty}^{x}s^{-1}\exp(\lambda\sigma\xi)K_1(\lambda\sigma s)\,d\xi, \tag{4.2}$$

with $s = (\xi^2 + y^2)^{\frac{1}{2}}$. Although p_{1p} satisfies the boundary condition (2.3), it does not represent the complete solution for p_1, because for $\alpha \neq \frac{1}{2}\pi$ p_{1p} is discontinuous across $y = 0$ when $x > 0$ (for $x < 0$ the integrand in (4.2) is bounded, so that p_{1p} is continuous). This may be seen as follows. Let Δ and δ be small positive numbers and assume that $x > 0$. One may write

$$p_{1p}(x, \Delta) - p_{1p}(x, -\Delta) \approx -\frac{\Delta\lambda\sigma\cos\alpha}{\pi}\left\{\int_{-\infty}^{-\delta} + \int_{\delta}^{x} + \int_{-\delta}^{\delta}\right\}s^{-1}\exp(\lambda\sigma\xi)K_1(\lambda\sigma s)\,d\xi. \tag{4.3}$$

As $\Delta\downarrow 0$ with δ fixed, the integrand in the first two integrals is bounded, so that their contribution goes to zero as $\Delta\downarrow 0$. By choosing δ sufficiently small the third integral can be arbitrarily closely approximated by

$$\int_{-\delta}^{\delta}(\lambda\sigma s^2)^{-1}\,d\xi = (2/\lambda\sigma\Delta)\arctan(\delta/\Delta), \tag{4.4}$$

where the asymptotic property

$$K_1(z) \approx 1/z \quad \text{for} \quad |z| \ll 1 \tag{4.5}$$

has been used. Hence

$$\lim_{\Delta\downarrow 0}\{p_{1p}(x, \Delta) - p_{1p}(x, -\Delta)\} = -\cos\alpha, \quad x > 0. \tag{4.6}$$

Thus p_{1p} has a jump $(-\cos\alpha)$ across the $+x$ axis. The solution for p_1 equals p_{1p} plus a harmonic function p_{1h} which vanishes at infinity, except perhaps downstream of the source (i.e. in the neighbourhood of the x axis), and has a jump $(+\cos\alpha)$ across the $+x$ axis. It will be shown that such a harmonic function does not exist. A harmonic function which has a jump equal to $\cos\alpha$ across the $+x$ axis is

$$p_{1h} = [\tfrac{1}{2} + \pi^{-1}\arctan(x/y)]\cos\alpha. \tag{4.7}$$

The function $p_{1p} + p_{1h}$ is continuous everywhere except at the origin, but it does not vanish at infinity, except for $\alpha = \tfrac{1}{2}\pi$. This deficiency may only be removed by the addition of a harmonic function which is singular only at the origin, and does not have a branch point. Such a function is the real or imaginary part of a complex function that can be represented by a Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n. \tag{4.8}$$

The only term of this series that is finite at infinity is the constant term a_0, which cannot be made equal and opposite to p_{1h} at infinity.

This proves that in two dimensions in an unbounded region a line-source solution of the Oseen–Boussinesq equations which vanishes at infinity does not exist, except possibly for the case $\alpha = \tfrac{1}{2}\pi$. It was shown in the preceding section that in three dimensions such a solution does exist. The situation is analogous to the Stokes paradox. This 'paradox' refers to the fact that the Stokes equations for low Reynolds number flow around a sphere have a solution, whereas a solution for the two-dimensional flow around a cylinder which leaves the flow at infinity unperturbed does not exist (see for example Van Dyke (1964, p. 152).) In the present situation the paradox is 'weaker' than the original Stokes paradox because there the velocity becomes infinite at infinity, whereas here it remains finite outside the wake, as is shown below.

As in three dimensions, the velocity field is of the form (3.8). The solution of the two-dimensional version of (3.3) in the case $\sigma = 1$ is

$$\chi_1 = (\lambda/\pi)\, r \exp(\lambda x)\, K_1(\lambda_r), \tag{4.9}$$

with $r^2 = x^2 + y^2$. Hence the rotational part of the velocity vanishes at infinity outside the wake for $\sigma = 1$, and presumably also for $\sigma \neq 1$. From (3.11) it follows that the potential part of the velocity is non-zero and finite at infinity.

It remains to be seen whether the Stokes-type paradox exhibited above is caused by the Oseen linearization or whether it is a property of the full Boussinesq equations.

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